

Relating spontaneous and explicit symmetry breaking in the presence of the Higgs mechanism

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Abstract

One common way to define spontaneous symmetry breaking involves necessarily explicit symmetry breaking. We study Quantum Field Theories extending the Standard Model, without anomalies.

We add explicit symmetry breaking terms to the Higgs potential, so that the spontaneous breaking of a global symmetry is a particular case of explicit symmetry breaking. This has the advantage that the classical Action indicates which symmetries are allowed to be spontaneously broken, without making non-perturbative assumptions that such symmetries are in fact spontaneously broken, and so such Action can be used in non-perturbative studies which may give support or not to such assumptions.

We study background fields and background symmetries: these are symmetries that despite they are already explicitly broken, can be still spontaneously broken.

Then we show that it is possible to study the Higgs potential without assuming that the local gauge $SU(2)_L$ symmetry is spontaneously broken or not (the spontaneous breakdown of a local gauge symmetry may not be possible). We also discuss the physical spectrum of extended Higgs sectors and the related custodial symmetry.

We clarify the assumptions under which a recent conjecture relating spontaneous and explicit CP-violation (charge-parity) is true/false; we relate explicit and spontaneous geometrical CP-violation (charge-parity).

We treat here the fields as real representations of the group of symmetries and show that this is consistent in Quantum Field Theory. If we know the (non-CP) symmetries of the Action, then there is a basis where CP is conserved if and only if the Action is Z_2 symmetric (i.e. there are no “phases” in the Action).

1 Introduction

There are several definitions of spontaneous breaking of global symmetries [1, 2], all are related with the existence of disjoint phases in a system¹. In the context of statistical mechanics [1],

¹Disjoint in the sense that a system cannot go through a phase transition by physically realizable operations. The converse is not true: there are topological phase transitions which do not involve symmetry breaking [3].

spontaneous symmetry breaking is often defined as a particular case of explicit symmetry breaking by a background field. Such definition was used before in the Standard Model [4] and in two-Higgs-doublet models (in the context of lattice simulations [5, 6]).

Let \mathcal{A} be an algebra of operators, let G be a group of global transformations $\mathcal{A} \rightarrow \mathcal{A}$.

The system's expectation value $\omega_{J,N}$ is a linear functional $\omega_{J,N} : \mathcal{A} \rightarrow \mathbb{C}$, J is a background field breaking the symmetry G , while N is the size of the system.

For finite size N , we assume that the system is well behaved with continuous expectation values² as a function of J , i.e. for any operator $A \in \mathcal{A}$ and any symmetry $g \in G$:

$$\begin{cases} \omega_{J,N}(A - g(A)) = 0, & \text{if } J = 0 \\ \lim_{J \rightarrow 0} \omega_{J,N}(A - g(A)) = 0 \end{cases}$$

Definition 1 (In statistical mechanics). The spontaneous symmetry breaking of G happens when there are finite expectation values breaking the symmetry G , for an arbitrarily small explicit symmetry breaking, i.e.

$$\lim_{J \rightarrow 0} \{ \lim_{N \rightarrow \infty} \omega_{J,N}(A - g(A)) \} \neq 0$$

for some $A \in \mathcal{A}$ and some $g \in G$.

The limits may not commute, because the (pointwise) limit of a convergent sequence of continuous functions is not necessarily continuous.

Other definitions in the context of statistical mechanics do not involve explicit symmetry breaking³, and are based instead on: a long-range order parameter which is the expectation value of a G -symmetric function $f(A)$ (e.g. the modulus $f(A) = |A|$) of an operator A which is translation invariant and breaks G ; or a conditional expectation value of some operator A given some condition $C = 0$ that breaks the symmetry; or a two-point correlation function with the points at an infinite distance from each other (related with boundary conditions⁴) [1]. It is widely accepted that these definitions should be all equivalent to Def. 1 (e.g. in the Ising model [1], although it does not seem easy to prove it because the systems with or without background field are physically different [8]).

When it comes to quantum non-abelian gauge field theories, the theories themselves lack a non-perturbative mathematical definition [9], so it is even more difficult to relate these different

²It is not strictly required that the expectation values are continuous for finite N to have spontaneous symmetry breaking [2], but the systems with local interactions (e.g. the Ising model or gauge theories) share this property. There are also systems where the thermodynamic limit makes no sense [7], requiring a more general definition of phase transition.

³Such definitions are not based on the existence of expectation values that explicitly break the symmetry, since that would not be possible by definition of the system's expectation value with $J = 0$. We prefer Def. 1 because it allows us to study the symmetries of the system at the Lagrangian level, independently of the particular Quantum Field Theory framework (e.g. perturbative/continuum or non-perturbative/UV-cutoff, scattering processes or bound-states), as we want to use several frameworks for phenomenology studies.

⁴There are yet more definitions based on an Hamiltonian formulation of statistical field theory [2] with the boundary conditions such as the initial time playing a key role. It is widely accepted that the definitions based on boundary conditions should be equivalent to Def. 1 [2, See Sec. 10.C].

definitions. By analogy with statistical mechanics⁵, these definitions make sense within the framework of quantum phase transitions [11, 12] (even at zero temperature). In the presence of the Higgs mechanism, there is yet another definition of spontaneous symmetry breaking, most common in the context of perturbation theory of the Electroweak interactions:

Definition 2 (Electroweak symmetry breaking). After a suitable perturbative non-abelian gauge fixing, the vacuum expectation value (vev) of the Higgs field is determined by one of the possible minima of the effective Higgs potential (calculated with perturbation theory). The symmetries broken by the Higgs vev are the spontaneously broken symmetries.

Def. 2 involves explicit symmetry breaking vevs since perturbation theory can only deal with small perturbations of the Higgs field, which is only guaranteed if the Higgs vev is non-null⁶. The determination of a non-null Higgs vev involves a mean-field approximation. Therefore, Def. 2 is consistent with Def. 1 under the non-trivial assumption that such mean-field approximation is appropriate, which is often not the case in statistical mechanics⁷.

However, the fact is that the perturbative predictions from the Electroweak theory seem to be a very good approximation to the existing experimental data in high-energy physics[22], and the (non-perturbative) lattice simulations so far support this picture [23–25] (also for two-Higgs-doublet models [5], but not for a grand-unified theory [26]). Therefore, it is important to confront these definitions, not only for formal reasons, but also for phenomenological reasons since non-perturbative lattice simulations [27] and the functional renormalization group [28] are becoming increasingly relevant in the studies of Electroweak physics and beyond, and are well established in Flavour physics and Quantum Chromodynamics (QCD).

There is a further ingredient to take into account [29]: a spontaneous breaking of local gauge symmetry without gauge fixing may be impossible in a gauge theory such as the Electroweak theory. The argument is based on the fact that local gauge transformations affect only a small sized system near each space-time point and so the two limits in Def 1 commute⁸. It can be argued that the Higgs mechanism avoids the presence of Nambu-Goldstone bosons precisely because the local gauge symmetry is not spontaneously broken [30, 31]. Many non-perturbative studies support this picture [32–35]. Moreover, there is a group-theory correspondence between gauge-invariant composite operators and the gauge-dependent elementary fields in the

⁵The correlation functions of quantum field theory can be defined as the Wick-rotation of correlation functions of a statistical field theory [10].

⁶When considering superselection sectors [13], we don't deal perturbatively with a null Higgs vev, we deal with a statistical ensemble of systems each with a non-null Higgs vev corresponding to one superselection sector and we study each system perturbatively.

⁷A simple example where the mean-field approximation predicts spontaneous symmetry breaking in disagreement with the exact solution is the one-dimensional Ising model [2, 12, 14]. On the other hand, there may be symmetries which we expect to be conserved, but due to yet unknown mechanisms in quantum field theory [15], are in fact spontaneously broken.

The classical problem of minimization of a polynomial [16–18] is a hard problem where topology is very useful [19, 20]. However, to find out which kind of topological transitions can entail a thermodynamic phase transition is still an open question [21].

⁸Under some assumptions on the analyticity of $\omega_{J,N}$. Note that since the quantum field theory is not well defined mathematically, it is hard to rigorously prove that Elitzur's theorem [29] is valid or that it is not valid.

Electroweak theory [31, 36] (also for two-Higgs-doublet models [6]).

The orbit space approach to the study of invariant functions[16] provides pure mathematical reasons why explicit and spontaneous symmetry breaking are necessarily related in the context of a (eventually non-renormalizable) potential of arbitrary order. However, such kind of relations were noted recently for the CP (charge-parity) symmetry in several multi-Higgs-doublet models and were summarized in the form of a conjecture [37, 38], but for renormalizable potentials which is intriguing. Previous intriguing relations in the same context were found earlier for a specific three-Higgs-doublet model [39].

In this paper we address four problems in the context of multi-Higgs-doublet models which are related, as we will see:

- check that the non-perturbative Def. 1 of spontaneous symmetry breaking is compatible with the usual assumptions of perturbation theory (Def. 2);
- how to study the Higgs potential and its phenomenological consequences without assuming spontaneous symmetry breaking of the gauge symmetry $SU(2)_L$;
- why the custodial symmetry is accidentally conserved in the Higgs potential of the Standard Model, and its relation with the physical spectrum;
- the relation between explicit and spontaneous symmetry breaking.

In Sec. 2 we state the assumptions we will make throughout the paper, reviewing background symmetries: these are symmetries that despite they are already explicitly broken, can be still spontaneously broken. In Sec. 3 we discuss explicit symmetry breaking, so that Def. 1 spontaneous symmetry breaking applies. Our assumptions and framework are compatible with the usual assumptions of Electroweak symmetry breaking (Def. 2), as we show in Sec. 4. In Sec. 5 we apply our formalism to the physical spectrum of extended Higgs sectors and the related custodial symmetry; also to several issues related with explicit and spontaneous CP violation. We conclude in Sec. 6.

2 Background symmetries of the classical Action

2.1 Background symmetries of the functional

Let \mathcal{A} be an algebra of operators, let G be a linearly reductive group⁹ of transformations $\mathcal{A} \rightarrow \mathcal{A}$, with G_b and $G_f \subset G$ normal subgroups of G .

Consider a G_f -symmetric linear functional $\omega : \mathcal{A} \rightarrow \mathbb{C}$, by definition all the symmetries conserved by ω are explicitly conserved by all correlation functions, independently of whether

⁹The class of linearly reductive groups includes all compact groups and the Lorentz group and its double covers that act on the spinors. Such class therefore covers all the groups that are relevant in the Standard Model and in many of its extensions.

the symmetries are spontaneously broken or not. That is, $\omega(A) = \omega(gA)$ for all $A \in \mathcal{A}$ and all $g \in G_f$.

The G_f -invariant operators are a representation space of the group G/G_f —we have the homomorphism $G \rightarrow G/G_f$ where G_f is the kernel of the homomorphism.

Consider now the functional ω_B depending on a G_f -invariant background field¹⁰ B , i.e. the expectation value $\omega_B(A)$ of the operator A is a (classical) function of the numerical values B and $gB = B$ for all $g \in G_f$. We can study the abelian subgroup associated with each generator of G/G_b separately. If some generator is broken then the group G/G_b is broken; if all are conserved then G/G_b is conserved. We can therefore assume without loss of generality that G/G_b is abelian and has a single generator c generating the abelian subgroup $C \subset G$.

In analogy with Def. 1, we say that G/G_b is a background symmetry of ω_B when $\omega_B(A) = \omega_{cB}(cA)$ for all G_b -invariant operator $A \in \mathcal{A}$, i.e. A verifies $\omega_B(A) = \omega_{gB}(gA)$ for all $g \in G_b$. We make no assumptions on whether ω_B is G_b -invariant or not. We do assume that the G_b -invariant operators are polynomials of the fields and of the background fields¹¹. Note that $g : \omega_B(gA) \rightarrow \omega_B(gA)$ affects only the fields (not the background fields) while the transformation $g : \omega_B(A) \rightarrow \omega_{gB}(A)$ affects only the background field, for any $g \in G$.

Suppose that c is conserved, then any background transformation $g \in G_b$ modifies the symmetry transformations as $a \rightarrow gag^{-1}$, that is $\omega_{gB}(gA) = \omega_B(A) = \omega_B(aA) = \omega_{gB}(gag^{-1}gA)$ (for a G_b -invariant operator A).

As consequence of the isomorphism theorems [46], the following groups are isomorphic $G/G_b \simeq (G/G_f)/(G_b/G_f)$ and the homomorphism $G \rightarrow G/G_b$ can be achieved in two steps: first $G \rightarrow G/G_f$ and then $G/G_f \rightarrow (G/G_f)/(G_b/G_f)$. This is important since we can build operators invariant under the background group G_b using only the operators invariant under the group of symmetries G_f that we constructed in a first step.

2.2 Spontaneous symmetry breaking of background symmetries

The CP symmetry is explicitly broken in the Standard Model, by the phase of the CKM matrix. Promoting such parameter to a background field B , we can still spontaneously break the CP background symmetry in extended Higgs sectors. We introduce a background field J which also breaks CP explicitly, and then we have spontaneous breaking of the CP background symmetry

¹⁰We only consider commuting (i.e. non-Grassmann) background fields. A spurion or (non-dynamical) background field enters in the definition of the Lagrangian but it is not a field of the Lagrangian. When calculating the observables, the background fields are replaced by numerical values. Such numerical values (and the usual fields) are a representation of a group of background symmetries of the classical Action, but there are no Noether's currents associated with such background symmetries if the numerical values are non-trivial. The observables are invariant under the action of the group of the background symmetries. See Ref. [40–42] for details and related studies.

¹¹There is a more general definition: G/G_b is a background symmetry of ω_B when for any $g \in G$ there is some $h \in G_b$ such that $\omega_B(A) = \omega_{ghB}(ghA)$ for all operator A . These two definitions are equivalent when G_b is a linearly reductive Lie group: there is then a finite algebraic basis of G_b -invariants parametrizing faithfully the G_b -orbit space [16].

For compact groups we can assume the operators to be smooth functions of the fields and of the background fields [43–45].

G/G_b when

$\lim_{J \rightarrow 0} (\omega_{J,B}(A) - \omega_{J,cB}(cA)) \neq 0$ for some G_b -invariant operator $A \in \mathcal{A}$ and c is the CP transformation, i.e. the generator of the CP group G/G_b .

We can consider analogous situations with other compact groups G . Using a second background field J_2 we could even study the spontaneous symmetry breaking of a symmetry that is already spontaneously broken via J . Therefore, the use of background fields allow us to address a wide class of problems.

For instance, the soft symmetry breaking terms are very useful for phenomenological applications [47]. These are quadratic terms of the Higgs potential, the corresponding parameters can be promoted to background fields, such that the symmetry which is softly broken is a background symmetry. We can therefore study spontaneous symmetry breaking in the context of softly broken symmetries.

2.3 Classical Action in Quantum Field Theory

The introduction of Grassmann (anti-commuting) variables to describe fermions in the classical Action, allows us to define in principle any quantum non-abelian gauge field theories by its classical Action, which is well defined mathematically [48]. While we could consider fermionic background fields, all the parameters of the classical Action (e.g. the Yukawa couplings) as well as the results of the correlation functions are commuting numbers and so it always suffices to consider commuting background fields, as we will do here.

However, the symmetries conserved by the classical Action may not be conserved by the path integral measure and so by the vacuum functional (i.e. by the full quantum system): we have then a quantum anomaly [49, 50]. Explicit symmetry breaking is different from quantum anomalies: in explicit symmetry breaking the classical action contains explicit symmetry breaking terms, such that when those terms are null both the classical action and the vacuum conserve the symmetry.

Since spontaneous symmetry breaking can be defined as a particular case of explicit symmetry breaking, it is also different from quantum anomalies. Background symmetries are a particular case of explicit symmetry breaking, as well.

We will study here the action of the group G on the classical Action and assume that the Action, the field content of the theory and the group G are chosen such that the path integral measure is G -invariant: i.e. there are no quantum anomalies. Such assumption is valid for the Standard Model [49] and many extensions. In any case, the determination of the symmetries of the classical Action is a first step towards the determination of the symmetries of the vacuum, therefore our results are also useful in the presence of quantum anomalies¹².

¹²The presence of quantum anomalies implies that the study of the symmetries of the vacuum must address also the path integral measure which still lacks a non-perturbative mathematical definition [9], and so can only be done using further assumptions appropriate for each particular quantum theory in separate.

2.4 The fields are real representations of the background symmetries

Holomorphic functions (i.e. $\frac{\partial f(z, z^*)}{\partial z^*} = 0$ where z^* is the complex conjugate of the complex variable z) are the central objects of study in complex analysis, which has many applications.

However, the classical Action is not an holomorphic functional: $\frac{\partial S(\phi, \phi^*)}{\partial \phi_x^*} \neq 0$ for any complex field ϕ .

Therefore, there is no a-priori advantage in the fields and background fields being complex vector spaces: $S(\phi, \phi^*) = S(\text{Re}(\phi), \text{Im}(\phi))$. For instance, the orbit space methods [16] are valid for real or complex vector spaces; the Wigner theorem is valid for unitary/anti-unitary representations on complex vector spaces as well as for real orthogonal (i.e. unitary) representations on real vector spaces [51].

We choose fields and background fields which are real vector spaces. The algebra of operators is a real algebra [52, 53], so that the operators are polynomials with real coefficients of the fields and background fields. Note that the fact that the algebra of operators is real does not prevent the vacuum expectation values (i.e. the probability amplitudes) from being complex. We have then two real linear functionals: the real part of the probability amplitude and the imaginary part of the probability amplitude. The vacuum expectation value of a complex operator can then be calculated as a linear combination of vacuum expectation values of real operators¹³.

The probability amplitude depends on the classical action. In the fermionic part of the Action, there is a difference between treating the fermions with commuting or Grassmann variables. For instance, $\phi^\dagger H \phi = \text{Re}(\phi^\dagger H \phi)$ but $\eta^\dagger H \eta = i \text{Im}(\eta^\dagger H \eta)$ for H a complex hermitian matrix, ϕ a complex commuting field and η a complex Grassmann field. That is, an hermitian matrix acting on commuting variables produces a real polynomial, but an hermitian matrix acting on Grassmann variables produces an imaginary polynomial, due to the anti-commuting properties of the Grassmann variables.

Therefore, the classical Action is in fact the direct sum of two real polynomials: the real part and the imaginary part of the classical Action. So, the probability amplitude depends on two real polynomials. Note that we cannot mix these two polynomials, for instance a polynomial involving only bosons must be hermitian and so it cannot be multiplied by a phase; if there would be no fermions, we would have only one hermitian polynomial. Therefore, the two real polynomials can not be seen (rigorously) as one complex polynomial in any case.

When the classical Action S transforms under CPT as $\text{Im}(S) \rightarrow -\text{Im}(S)$ and $\text{Re}(S) \rightarrow \text{Re}(S)$, then we assume that the probability amplitude A transforms under CPT as $\text{Im}(A) \rightarrow -\text{Im}(A)$ and $\text{Re}(A) \rightarrow \text{Re}(A)$. The probability is given by $(\text{Re}A)^2 + (\text{Im}A)^2$ and thus we say that CPT is conserved.

Note that the standard notation is more intuitive, using complex numbers in the path

¹³In a more elaborate treatment we can define momentum spaces in the real fields (both bosonic and fermionic)[54] and so in fact we do not evaluate vacuum expectation values of complex operators. However, for the purposes of this work it suffices to know that we can calculate the vev of a complex operator as a linear combination of vevs of real operators, leaving the question of whether it is necessary to evaluate vevs of complex operators to another opportunity.

integral measure and in the probability amplitude. But we have shown that despite intuitive, such notation does not imply that the fields and the algebra of fields is complex. On the contrary, the standard approach is compatible with the choice of real fields and a real algebra of fields, which is expected since the classical Action really is a classical functional in the sense of Classical Field Theory and there is no notice that complex fields are indispensable in Classical Field Theory [48].

But still we could instead (in principle) treat the fields as complex representations, but the irreducible representations would not match the standard classical ones. This is because the imaginary number appearing in the path integral measure is CP -symmetric and not T -symmetric, while the imaginary number appearing in the classical irreducible complex fields is the one from the gauge transformations which is T -symmetric and not CP -symmetric.

For instance, the standard classical Higgs field is T symmetric, if we want to treat it as a complex representation with the imaginary number appearing in the path integral measure we would need to complexify it and then we would have two irreducible $SU(2)_L$ Higgs doublets in the Standard Model, corresponding to two possible relations between the two different commuting imaginary numbers $i_1 = \pm i_2$. This would be transferring the well known problem of the negative energy states from perturbation theory to the classical Action. While in perturbation theory such problem is solved by using the Feynman propagator (which contains the correct projector to the positive-energy state) and the associated Feynman-Stueckelberg interpretation of anti-particle [55], at the level of the classical Action the problem is known since the appearance of the Dirac equation [56] and no good solution was found so far in the context of the classical Action, to the knowledge of the author.

Sometimes, optionally we can still use complex representations with the imaginary number from the gauge transformations. On the one hand, there is the advantage that complex irreducible representations of the group $G \times H$ are a direct product of complex irreducible representations of G and of H , which is not the case for real irreducible representations. On the other hand, we cannot do it always since we cannot define all linear operators, some of these operators have important experimental consequences, for instance the approximate custodial symmetry can only be defined when the Higgs field is a real representation of $SU(2)_L$ [22, 57, 58] or the Majorana condition on fermions cannot be defined [59]. Moreover, this is not the standard approach [60–62] since then the CP transformation would be anti-unitary¹⁴, and thus it is not an approach which is well developed in Quantum Field Theory and usually it is not what authors mean by using complex representations.

2.5 Effective Action and ultra-violet incomplete theories

We assume that the classical Action is a real polynomial of arbitrary order on the fields. From the point of view of classical field theory there is no reason to limit the order of the polynomial.

¹⁴Anti-unitary as in reference [63]). That this is not the standard approach was recognized in Ref. [64] which shares a common author with Ref. [63].

When taking into account the quantum effects, then we are working in the framework of an effective field theory, without making assumptions about the ultra-violet completion of the theory¹⁵. Surely, we require the classical Action to be such that the quantum theory is predictive enough and valid up to an interesting energy scale: implying that higher order interactions should be fewer and much smaller [70], but not necessarily a fourth order (or any other limit on the order of the) polynomial. For instance, the two-Higgs-doublet model can be formulated as an effective field theory [69].

3 Explicit symmetry breaking

3.1 Necessary and sufficient conditions

Without knowing much about our system, we can state necessary and sufficient conditions for explicit symmetry breaking by a background field J , with arbitrary numeric values but a representation (not necessarily a vector space, it may be just a set of numeric values) of the group G . We will call J the source field, to distinguish it from the remaining background fields. As in the previous section, G is a linearly reductive group, with G_b and $G_f \subset G_b$ normal subgroups of G ; without loss of generality, we assumed G/G_b is abelian and has a single generator c generating the abelian subgroup $C \subset G$.

The background symmetry is by definition always explicitly conserved in the absence of the source field. Therefore, when we refer to the explicit breaking of a background symmetry we mean in the presence of the source field. We assume that the numerical values of the G_f -invariant background fields conserve the generator c , thus the group C is conserved by the G_f -invariant classical Action.

There are then 2 different possibilities for the source field:

1) J may break the background symmetry G/G_b For at least some numerical values of the source field, the G_b -invariants involving only the source field break G/G_b . This implies that there is an Action which breaks G/G_b , it may be renormalizable or not (E.g.: for one complex

¹⁵The claim that a quantum field theory is ultra-violet complete (i.e. renormalizable) just because the classical Action is a fourth order polynomial has some problems: without an ultra-violet cutoff (or some alternative regularization), quantum non-abelian gauge field theories still lack a non-perturbative mathematical definition [9]; the perturbative approach to the Standard Model is based on the $\lambda\phi^4$ quantum theory (mexican hat potential), but in the $\lambda\phi^4$ quantum theory the (non-perturbatively) renormalized coupling λ seems to be necessarily null (trivial) [10, 65], the triviality can be avoided with an ultra-violet cutoff and an upper bound for the Higgs mass [66]; the advantage of a logically consistent ultra-violet complete theory over one incomplete theory would be to explain the meta-stability of the Standard Model in face of the present experimental data [28, 67] and quantum gravity (at the Planck scale), no such theory is yet known.

Note that extensions to the Standard Model often have so much degrees of freedom that either we set the non-renormalizable interactions to zero or not, they are anyway phenomenologically relevant. E.g. the renormalizable two-Higgs-doublet model has enough degrees of freedom to emulate a standard Higgs sector with free couplings at the LHC (ignoring the non-LHC constraints) [68]; while the non-renormalizable two-Higgs-doublet model is also relevant [69].

We can have in principle constant fields without space-time dependence, for which renormalizability imposes no restriction on the order of the polynomial. Or a time-independent problem, where the restriction is different from fourth order.

scalar field, $G_f = Z_6$ and $G = U(1)$ we need a non-renormalizable potential to break $U(1)$ while conserving Z_6). It may happen that all numerical values of the source field break G/G_b .

Then there is $J = J_0$ that breaks the true symmetry C/G_f and the breaking depends on J_0 .

2) J necessarily conserves the background symmetry G/G_b For any numerical values of the source field, the G_b -invariants involving only the source field conserve G/G_b . Therefore, for any source field J there is $h \in G_b$ such that $J = hJ_0$ and J_0 conserves c (i.e. $cJ_0 = J_0$). Then $hch^{-1}J = J$ and the background symmetry G/G_b is conserved.

The question now is whether there is some J_0 (with $cJ_0 = J_0$) such that $[C, G_b]/G_f J_0 \neq G_f J_0$, i.e. whether there is some $h \in G_b$ such that $[c, h]G_f J_0 = hch^{-1}c^{-1}G_f J_0 \neq G_f J_0$, where $[c, h] = chc^{-1}h^{-1}$.

2.A) J_0 necessarily conserves the background symmetry $[C, G_b]/G_f$ Since we can write any J as $J = hJ_0$ with $h \in G_b$ and $cJ_0 = J_0$, we have then that all J conserve c ($cG_f J = chG_f J_0 = hcG_f J_0 = hG_f J_0 = G_f J$) and explicit breaking of C/G_f by the source field is not possible.

2.B) J_0 may break the background symmetry $[C, G_b]/G_f$ There is then some $h \in G_b$ and some J_0 such that $chG_f J_0 \neq hcG_f J_0$. But $(hch^{-1})hJ_0 = hJ_0$.

So the explicit symmetry breaking of the true symmetry C/G_f is allowed for $J = hJ_0$, while the background symmetry G/G_b is necessarily conserved. The transformation h can be transferred to the background fields (leaving hJ_0 invariant under (hch^{-1})) and so we have explicit breaking of G/G_b independent of the numerical values of hJ_0 (the only requirement is $chG_f J_0 \neq hcG_f J_0$).

4 Spontaneous symmetry breaking due to the Higgs potential

4.1 Assumptions

The only difference with respect to the usual perturbative expansion is that we only evaluate vevs of $SU(2)_L$ -invariant operators so we do not assume that $SU(2)_L$ is spontaneously broken. We consider a G/G_b -invariant and G_f -invariant potential. The subgroups $G_b, G_f \subset G$ and $SU(2)_L$ -gauge (with $SU(2)_L \subset G_f$) are normal subgroups of G which is a compact group. We also assume that G/G_f is a group of global transformations¹⁶.

In analogy with Def. 2, to study the Higgs potential and in particular the global symmetries which are spontaneously broken or not, we assume that:

¹⁶Since the $U(1)_Y$ gauge symmetry is abelian and there are no Gribov-Singer ambiguities for abelian gauge fixing (unlike for a non-abelian gauge symmetry such as $SU(2)_L$), we can unambiguously fix the local gauge with a gauge-fixing local term and deal only with the $U(1)_Y$ global symmetry. Note that we explicitly mention $SU(2)_L$ because of the Standard Model, but our formalism is valid for any compact G_f .

- we fix the local gauge and calculate the effective potential, using the standard methods: perturbative expansion in the number of loops and renormalization group running [70–72]);
- We then assume that the vevs of operators are given by the usual perturbative expansion, which is an expansion in 1) the Higgs field around one point $\frac{v}{\sqrt{2}}\phi_0$ (constant in space-time and $\phi_0^\dagger\phi_0 = 1$) for which the effective potential has an absolute minimum; and 2) the couplings of the interactions¹⁷;
- whenever there are two or more G_f -orbits minimizing the Higgs potential, then there is spontaneous symmetry breaking of the global symmetry G/G_b if these G_f -orbits are related by G .
- we assume that the non-perturbative vevs of operators conserve the same symmetries as the vevs calculated in perturbation theory, i.e. the symmetries conserved by the classical Action and the reference point [70].

These are non-trivial assumptions, requiring that the mean-field approximation describes spontaneous symmetry breaking correctly. In the Standard Model, the $SU(2)_L$ -gauge orbit minimizing the Higgs potential is unique and therefore there is no experimental evidence in the context of Electroweak physics, that these assumptions relating spontaneous symmetry breaking with non-unique $SU(2)_L$ -gauge orbits are valid. Thus more studies, simulations and experimental data are required to support these assumptions [5, 6]. Even if these assumptions are valid, it is still not an easy task to determine if some symmetry is spontaneously broken or not by the radiative corrections, if it is conserved at tree-level [77–79]. Note however that the radiative corrections do conserve the background symmetries explicitly, in the absence of quantum anomalies in the measure of the path integral.

The electromagnetic symmetry $U(1)_{em}$ is the representation of the $U(1)_Y$ gauge symmetry in the $SU(2)_L$ -invariant operators (see Sec. 4.3) and we can treat it as a global symmetry after $U(1)_Y$ local gauge fixing. Therefore, under our assumptions the $U(1)_{em}$ symmetry can also be spontaneously broken like all other global symmetries if there are two $SU(2)_L$ -gauge orbits

¹⁷The usual perturbative expansion is an expansion in the couplings with the mass of the W boson kept finite ($M_W = gv/2$) therefore it is also an expansion for large Higgs vev.

The vevs of the $SU(2)_L$ -gauge-invariant operators are the physical observables if the $SU(2)_L$ gauge symmetry is not spontaneously broken, as it seems to be the case [30, 31, 36]. In the context of the perturbative formulation of Electroweak theory, there are already studies of the (multi-)Higgs potential based on $SU(2)_L$ -invariant bilinears of the Higgs field [58, 73–75].

The local gauge fixing is perturbative with a local term and in a suitable gauge [36, 76] (such as the usual gauges used in perturbation theory), we assume that the (non-perturbative) Gribov-Singer ambiguities do not affect our results. The reference point is constant in space-time with respect to the chosen $SU(2)_L \times U(1)_Y$ local gauge.

minimizing the Higgs potential related by a $U(1)_{em}$ transformation¹⁸.

Under these assumptions, we have to solve a classical (but still non-perturbative) problem of minimization of a polynomial invariant under a group of symmetries [17, 18].

4.2 Procedure to add explicit symmetry breaking terms to the Higgs potential

If there is spontaneous symmetry breaking of C/G_f under the previous assumptions, then we modify the tree-level Higgs potential adding an infinitesimal G_f -invariant polynomial (defined by a source field J) and such that the chosen G_f -orbit minimizing the modified effective potential is not related by G to another absolute minimum¹⁹. Then the modified (tree-level) potential explicitly breaks G/G_b . If perturbation theory is correct there are finite vevs breaking G/G_b in the limit that the modified potential converges to the Higgs potential and so there is spontaneous symmetry breaking. However, our procedure is still valid in case perturbation theory fails, since what we determined were the symmetries of the (infinitesimally) modified tree-level potential.

Now we show that it is always possible to modify the potential by an infinitesimal term such that its minimum is the one we want. Note that we can make the added term smaller than any radiative correction and so the radiative corrections to the term are neglected in the perturbative expansion.

We take the numerical values of the source field $J = \epsilon\phi_0$ proportional to the numerical values of a Higgs field ϕ_0 with N real components corresponding to the absolute minimum of the Higgs potential.

The important point is that the orthogonal group $O(N)$ (G is compact) acts on the $\mathbb{R}^{\otimes n}$ tensor space with unitary operators. There is a one-to-one correspondence between isomorphisms of a vector space V to all of V^* and nondegenerate bilinear forms on V . Since $O(N)$ is compact, we can find a scalar product on V that makes the representation unitary.

For $n = 1$, we have the basis e_j and the bilinear form $\langle e_j, e_k \rangle = \delta_{jk}$. Such form makes the representation unitary. For arbitrary n , we have the basis $e_{j_1} \otimes \dots \otimes e_{j_n}$ and the $2n$ -linear form $\langle e_{j_1} \otimes \dots \otimes e_{j_n}, e_{k_1} \otimes \dots \otimes e_{k_n} \rangle = \frac{1}{n!} \sum_{\sigma} \delta_{j_1\sigma(k_1)} \dots \delta_{j_n\sigma(k_n)}$. Such form makes the representation of $O(n)$ unitary.

¹⁸We are not dependent on these assumptions to determine what would happen if the $SU(2)_L$ -gauge orbit minimizing the Higgs potential breaks the $U(1)_{em}$ generator: the photon would become massive due to the abelian Higgs mechanism—there are theoretical arguments [80] and also experimental evidence from superconductivity where the abelian Higgs mechanism also happens. It would not depend on the $U(1)_Y$ gauge-fixing and would not imply spontaneous breaking of the local gauge $U(1)_{em}$ [2]. The $U(1)_Y$ gauge-fixing merely allows us to simplify the study by treating the $U(1)_Y$ symmetry and the remaining global symmetries in the same consistent way, which is particularly useful to interpret the results of non-perturbative lattice studies where $U(1)_Y$ is not a local gauge symmetry (reducing computation time) [23–25].

¹⁹Note that the reference point may not be unique, then we still make an arbitrary choice but not one which breaks G .

Also, it does not suffice to explicitly break the symmetry in the classical Action without affecting the minimum of the modified effective potential, otherwise it would be possible to have vevs breaking the symmetry G even when the explicit symmetry breaking term is exactly null which would be inconsistent with Def. 1).

Therefore, suppose we have a polynomial strictly of order 2 in ϕ . It can be written using the inner product of the tensors $p^{jk}e_j \otimes e_k$ and $\phi^j \phi^k e_j \otimes e_k$

Now we want to find a G_f -invariant tensor whose maximum occurs at $\phi = \phi_0$ (for $G_f \phi_0$ breaking C).

There is a basis of G_f -invariant tensors, which is given by $\rho_a^{jk} e_j \otimes e_k$. The ρ_a are chosen such that $\langle \rho_a, \rho_b \rangle = \delta_{ab}$, using the Gram-Schmidt process. Note that the basis ρ_a is complete in the space of G_f -invariant 2nd order tensors, but it is incomplete in the space of 2nd order tensors. We can complete it nevertheless. The first n_f components are G_f -invariant, the remaining $n^2 - n_f$ components are not G_f -invariant.

Then, $\rho_a^{lm} \rho_a^{jk} = \frac{\delta_l^j \delta_m^k + \delta_l^k \delta_m^j}{2}$ with a running until n^2 . We then write $\Phi^a = \langle \rho_a, \phi_0 \otimes \phi_0 \rangle$ and $\Phi_f = \sum_{a=1}^{n_f} \Phi^a \rho_a$ is a G_f -invariant tensor.

Then, we can write any tensor as $T = cR\Phi_f$ where $c > 0$ is a normalization factor and R is a $O(n^2)$ rotation. Then, the R which maximizes the inner-product $\langle T, \Phi_f \rangle$ is $R = 1$. Of course, not all R is meaningful. R should be such that it can be written as a $O(n) \otimes O(n)$ rotation. But since the representation of $O(n)$ is unitary, then $O(n) \otimes O(n) \subset O(n^2)$ and so $R = 1 \in O(n) \otimes O(n)$ is a valid rotation. Also, $T = c\Phi_f$ is a G_f -invariant tensor which can be written as $T = \sum_{a=1}^{n_K} \langle \rho_a, \phi \otimes \phi \rangle \rho_a$.

Therefore, the Higgs field minimizing the polynomial of order 2 $V_2 = -\sum_{a=1}^{n_K} \langle \rho_a, \phi \otimes \phi \rangle \langle \rho_a, \Phi_f \rangle$ is ϕ_0 . At each order we can do the same and so we conclude that we can always add an infinitesimal G_f -invariant polynomial (defined by a source field $J = \epsilon \phi_0$) such that the chosen G_f -orbit minimizing the modified potential is not related by C to another absolute minimum.

Note that since the potential V has a background symmetry G then $V_B(\phi) = V_{cB}(c\phi)$ and also the reference point transforms under G covariantly with respect to the background fields [42], i.e. $W_{B,J}(\phi) = W_{gB,gJ}(g\phi)$ for all $g \in G$. By construction, a background transformation $c \in G/G_b$ is conserved (i.e. it is not spontaneously broken) if and only if c is conserved by the modified Higgs potential $W_{B,J}(\phi) = W_{chB,J}(ch\phi)$ for some $h \in G_b$. Therefore, the numerical values of background fields and source fields, suffice to determine the spontaneously broken symmetries.

4.3 Compatibility with Electroweak symmetry breaking

In the procedure described in Sec. 4.2, we are not assuming that the $SU(2)_L$ -gauge symmetry is spontaneously broken, but we are not assuming that it is not spontaneously broken either; the assumptions made in Sec. 4.1 are compatible with further assumptions on gauge symmetry breaking, and they are suitable for studies looking for evidence of the spontaneous breaking of the $SU(2)_L$ -gauge symmetry—e.g. comparing perturbative predictions from vevs of gauge-invariant/dependent operators with experimental results and with non-perturbative studies.

After (perturbative) local gauge fixing and neglecting Gribov-Singer ambiguities, we can treat the $SU(2)_L$ as a global symmetry. Consider the most general Higgs potential V which has a background symmetry G and $SU(2)_L$ is a true symmetry. Assuming spontaneous symmetry

breaking of $SU(2)_L$, we have the Higgs vev $\Omega_{B,\epsilon\phi_0}(\phi) = \frac{v}{\sqrt{2}}\phi_0$, where Ω is the $SU(2)_L$ -gauge-dependent vacuum functional²⁰.

In analogy with Sec. 4.2, $g \in G/SU(2)_L$ is conserved by $\Omega_{B,\epsilon\phi_0}$ when $\Omega_{B,\phi_0}(A) = \Omega_{gB,\phi_0}(gA)$ for all operator A whose vev conserves $SU(2)_L$. In Def. 2, the Higgs vev is enough to determine the spontaneously broken symmetries, then the $SU(2)_L$ -invariant operators which only depend linearly on the Higgs field are enough to determine the spontaneously broken transformations $g \in G$. That is, we have to evaluate the vevs of the $SU(2)_L$ -invariant operators $\Psi^\dagger\phi$, where ϕ is the Higgs field and Ψ is a polynomial of the background fields and ϕ_0 .

Since $\epsilon\frac{v}{\sqrt{2}}\phi_0$ is the only background field which is not $SU(2)_L$ -gauge invariant, then a non-infinitesimal A depends on the reference point $\frac{v}{\sqrt{2}}\phi_0$ via $SU(2)_L$ -invariants only—the invariant tensors of $SU(2)_L$ are the Kronecker delta and the Levi-Civita tensor, which involve bilinears only.

Therefore, a transformation $g \in G$ is broken (spontaneously or explicitly) in the gauge-dependent vacuum functional Ω if and only if it is broken in the gauge-invariant vacuum functional of Sec. 4.2.

We conclude that it is possible to study the Higgs potential without making assumptions on whether the $SU(2)_L$ -gauge symmetry is conserved or spontaneously broken.

4.4 Relating spontaneous and explicit symmetry breaking

Consider a G/G_b -invariant and G_f -invariant potential. We assume without loss of generality (see Sec. 2) that G/G_b is abelian with generator c generating the abelian group C .

We identify the vev of the Higgs field with the source field.

If explicit breakdown of the background symmetry G/G_b is possible, then in the potential $V(\phi) = -\phi^\dagger\phi + (\phi^\dagger\phi)^2$ we can choose any direction for the reference point, including one direction which breaks the background symmetry G/G_b .

If explicit breakdown of the background symmetry G/G_b is not possible, then $G_f\phi = hG_f\phi_0$, for every Higgs field ϕ and some $h \in G_b$ and some Higgs field ϕ_0 verifying $CG_f\phi_0 = G_f\phi_0$.

If explicit symmetry breaking of C/G_f is not possible then there is no ϕ_0 which breaks G_b/G_f so that $\phi = \phi_0$ and spontaneous symmetry breaking is not possible either. Now we assume that there is ϕ_0 which breaks G_b/G_f .

Then, we can promote the elements of the group G_b to fields and then $\phi = h\phi_0$ is a product of two fields $h \in G_b$ and ϕ_0 . Then for all $c \in C$, we have $cG_f\phi = c(hG_f\phi_0) = (chc^\dagger)(cG_f\phi_0) = (chc^\dagger)\phi_0$. That is, c acts on h as $h \rightarrow chc^\dagger$ and leaves ϕ_0 invariant. The linear space structure of ϕ is broken, however it is well known that the groups do act in the elements of (other) groups not only on linear spaces; moreover this factorization is analogous to the use of polar coordinates in the treatment of the Higgs mechanism both perturbatively in the unitary gauge [57] as well as non-perturbatively in the lattice [32–35]. Also, this example can be made renormalizable

²⁰For instance, we can add the explicit symmetry breaking term $U = -\frac{v}{\sqrt{2}}\phi^\dagger\phi_0 + \phi^\dagger\phi$ to the Higgs potential $V + \epsilon U$, so that the minimum of the potential is unique and given by $\frac{v}{\sqrt{2}}\phi_0$.

by choosing ϕ without space-time dependence. Therefore such coordinates are allowed both in classical and quantum field theory.

We downgrade ϕ_0 to a parameter, constant in space-time such that ϕ_0 breaks G_b/G_f . Then, to break C spontaneously it suffices to find a function of G_b/G_f and C -symmetric whose minimum breaks C . Then the classical potential $V(\phi) = -\bar{p}^\dagger(\phi)\bar{p}(\phi)$ spontaneously breaks K , where $\bar{p} = \int_C dg g p(\phi) = (\int_C dg g h' g^\dagger) p(\phi_0)$ is the average of an irreducible representation of G/G_f which is not G_b -invariant (i.e. $h' \in G_b/G_f$ and p is a direct sum of G_f -invariant polynomials of ϕ with dg being the Haar measure of C). Note that $\bar{p}^\dagger(\phi)\bar{p}(\phi) = \int_{C^2} dg dk p^\dagger(\phi_0) k h'^\dagger k^\dagger g h' g^\dagger p(\phi_0)$ is an integral over inner products of misaligned vectors with the same size, thus it is smaller than the integral over inner products of aligned vectors with the same size which equals $\bar{p}^\dagger(\phi)\bar{p}(\phi)$ when h' commutes with C/G_f .

We have shown that if explicit symmetry breaking of C/G_f is allowed, then spontaneous symmetry breaking is allowed, i.e. there is always a potential which conserves the background symmetry G/G_b but breaks C/G_f spontaneously (assuming that the mean-field approximation is valid).

5 Applications

5.1 Custodial symmetry and the physical spectrum

As it was mentioned in Sec. 4.1, we can consider the $U(1)_Y$ gauge symmetry as a global symmetry. In this section, we will not consider the $U(1)_Y$ gauge symmetry at all, local or global. There are good reasons for this: for efficiency, the lattice simulations of Electroweak physics usually do not include the photons; the custodial accidental symmetry (which includes the global $U(1)_{em}$ and CP) plays an important role in Electroweak physics [22, 57], despite that it is broken by the $U(1)_Y$ gauge field; this apparent redundancy is important to predict correctly the number of pseudo-Goldstone bosons after spontaneous symmetry breaking in the multi-Higgs doublet models [58, 73, 74].

The group of background symmetries G includes $G_f = SU(2)_L$ as a normal subgroup, but the outer automorphisms of $SU(2)_L$ are trivial so $G = G_g \times SU(2)_L$. Therefore $G_g = G/G_f$ commutes with the generators of the $SU(2)_L$ gauge group.

For one Higgs-doublet, the background symmetry is $G_g = Sp(1)/Z_2 \simeq SU(2)_R/Z_2$. The only Higgs bilinear is $\phi^\dagger\phi$. Therefore under the assumptions of Sec 4.1, the custodial symmetry cannot be explicitly broken or spontaneously broken since there is only one $SU(2)_L$ -orbit.

On the other hand, for more than one Higgs doublet, there is by construction more than one $SU(2)_L$ -orbit and we can have the Pauli matrices $i\sigma_j$ corresponding to the custodial $SU(2)_R$ generators appearing in the Higgs bilinears.

We introduce now a $SU(2)_L$ -invariant unitary matrix R , related with the reference point $\varphi_0 = R^\dagger\phi_0$ such that only the first $SU(2)_L$ doublet is non-null $(\varphi_0)_k = \delta_{k1}(\varphi_0)_1$. We can then express the Higgs field as $\varphi = R^\dagger\phi$.

Suppose that the quadratic part of the G -invariant potential V is given by $\phi^\dagger Y \phi$ and the quartic part is $(\phi^\dagger \otimes \phi^\dagger) Z (\phi \otimes \phi)$ ²¹. We have then the parameters $\mathcal{Y} = R^\dagger Y R$ and $\mathcal{Z} = (R^\dagger \otimes R^\dagger) Z (R \otimes R)$ and the reference point φ_0 . We call this basis the Higgs basis.

We assume now without loss of generality, that $i\sigma_j \varphi_0 = i\tau_j \varphi_0$ after gauge-fixing, where $i\tau_j$ are the $SU(2)_L$ generators. From the Higgs field φ we can also form $SU(2)_L$ -invariant operators, in particular the operators $\varphi_1^\dagger \varphi_k$ and $\varphi_1^\dagger i\sigma_j \varphi_k$ expand to the elementary fields in leading order of the expansion around the reference point [36] and therefore correspond to the physical Higgs bosons appearing in the spectrum of the model, except for the would-be goldstone bosons $\varphi_1^\dagger i\sigma_j \varphi_1$ since $\varphi_1^\dagger i\sigma_j \varphi_1$ is null due to the fact that the $i\sigma_j$ are skew-hermitian 4×4 real matrices. Also $\varphi_1^\dagger D_\mu i\sigma_j \varphi_1$ expands to the $SU(2)_L$ gauge fields and $\varphi_1^\dagger \Psi$ and $\varphi_1^\dagger i\sigma_j \Psi_L$ expand to an elementary left-handed fermion field, where D_μ is the covariant derivative involving the $SU(2)_L$ gauge fields and Ψ_L is a fermionic $SU(2)_L$ doublet in the Higgs basis²².

5.2 Charge-Parity (CP) symmetry

Following Sec. 5.1, we now consider the global $U(1)_{em}$ symmetry as a symmetry of the potential. Unlike $SU(2)_L$, there is one outer automorphism of $U(1)_{em}$: Z_2 related with the CP transformation. Note that Z_2 has no outer automorphisms. Therefore, we have $G = ((G_g \times G_f) \rtimes Z_2)$ where $G_f = U(1)_{em} \times SU(2)_L$ and $G/G_f = G_g \rtimes Z_2$.

The group Z_2 in $G_g \rtimes Z_2$ is generated by the CP (charge-parity) transformation given by $\varphi \rightarrow -i\tau_1 i\sigma_1 \varphi$. The generator of $U(1)_{em}$ is given by $\varphi \rightarrow (i\sigma_3 - i\tau_3) \varphi$. So the imaginary unit corresponds to $i\sigma_3$ and the CP transformation involves a complex conjugation.

Note that as in Sec. 5.1, $i\sigma_j \varphi_0 = i\tau_j \varphi_0$ after $SU(2)_L$ gauge fixing, where $i\tau_j$ are the $SU(2)_L$ generators and φ_0 is the reference point.

Neutral vacuum If we assume that $U(1)_{em}$ is a true symmetry in the Higgs basis, then the parameters of the Higgs potential \mathcal{Y}_{kl} and $\mathcal{Z}_{km \ln}$ are complex tensors—i.e. they commute with $i\sigma_3$.

All phases in the Higgs basis come from background fields, since the $Z_2 = G/G_g$ (complex conjugation in the Higgs basis) background symmetry is not spontaneously broken. In this sense, CP violation is always determined by the background fields and associated background symmetry G_g .

5.3 Rephasing symmetries

We now analyze some models related with a conjecture relating explicit and spontaneous CP violation [37, 38].

²¹For simplicity we consider a potential up to fourth order, but we could consider more orders here.

²²See [6] for more details on how to add the $U(1)_Y$ gauge field and the Yukawa couplings, in the case of the two-Higgs-doublet model—the generalization for any other model with $SU(2)_L$ gauge symmetry is straightforward in the Higgs basis.

If G_f is abelian and it is a subgroup of $U(n) \times SU(2)_L$, then it commutes with the group of rephasing transformations of the Higgs fields $G_b = U(1)^{n-1} \times U(1)_{em} \times SU(2)_L$. Then any neutral Higgs field can be written as $\phi = e^{i \sum_{k=1}^{n-1} \theta_k} \phi_0$ with ϕ_0 real and so verifying $c\phi_0 = \phi_0$ where c is the complex conjugation (related with the CP symmetry) and θ_k are phases parametrizing the group $G_b/(U(1)_{em} \times SU(2)_L)$.

Therefore, the background symmetry $G/G_b = Z_2$ cannot be explicitly broken by a source field which is a copy of a neutral Higgs field. It also cannot be spontaneously broken by a neutral vev of the Higgs field.

Consider now the potential restricted to a neutral Higgs and the phases θ_k of a basis parametrizing $U(1)^{n-1}$. If some phase θ_k appears in more than one term of the potential, then we can have explicit symmetry breaking of the CP symmetry. But if each phase θ_k appears at most in one term, then we cannot have explicit CP violation; also in such case, spontaneous CP violation (at tree level) is limited by the critical points of $\cos(\theta_k)$ [40].

However, despite the fact that the results are based in a symmetry G_f , these may not hold when radiative corrections are taken into account. Such radiative corrections conserve the explicit symmetry but may introduce extra terms which will change the condition for the critical points. That is already the case for the two-Higgs-doublet model with a Z_2 flavor symmetry [78, 79].

If G_f is non-abelian, but the phase dependent part of the potential has one and only one arbitrary parameter (the remaining parameters are consequence of G_f) then the phases are not arbitrary. Such is the case of an order-4 potential of a 3-Higgs-doublet model symmetric under $((A_4 \times U(1)_{em}) \rtimes Z_2) \times SU(2)_L$ or $((\Delta(54)/Z_3) \times U(1)_{em}) \rtimes Z_2 \times SU(2)_L$. But in the case of A_4 and $\Delta(54)/Z_3$, because the background group orbit does not absorb all the phases (i.e. the G_b -invariants are not CP-invariants, where G_b contains A_4 or $\Delta(54)/Z_3$ as a normal subgroup), the potential $V(\phi) = -\phi^\dagger \phi + (\phi^\dagger \phi)^2$ allows us to choose an arbitrary minimum which may break the background CP symmetry. Even assuming a strictly A_4 or $\Delta(54)/Z_3$ -invariant potential, there are the radiative corrections to take into account (since they affect the two-Higgs-doublet model case, they should also affect these cases as well [78, 79]).

Considering non-renormalizable potentials may also increase the number of phase-dependent terms (also for the case of an abelian symmetry), so we have to recount them and check the consequences. For instance, it is known that for the $\Delta(54)/Z_3$ -symmetry the requirements to obtain discrete phases of the Higgs vev (at tree level) in a non-renormalizable potential [81] have to change with respect to the requirements in a renormalizable potential [42].

We conclude that these discrete phases resulting from the minimization conditions require several requirements beyond the symmetry imposed, such conditions are not guaranteed to hold when radiative corrections are taken into account. So, we need to make the full calculations to ensure that these discrete phases survive.

Note however that the examples present in the conjecture [37, 38] as well as the example of spontaneous geometrical CP violation (the $\Delta(54)/Z_3$ case [39], also treated next) were most

helpful for this work.

5.4 Explicit geometrical CP violation

One example is the so called explicit geometrical CP violation [82, 83], where CP stands for charge-parity.

Geometrical CP-violation involves calculable phases [39, 42, 63, 83, 84].

The idea of spontaneous geometrical CP-violation arose in a three-Higgs-doublet model, with a $\Delta(54)$ -symmetric Higgs potential which is a polynomial of fourth order. There is also explicit geometrical CP-violation [82, 83] which shows that the root of geometrical CP-violation is not the process of minimization of a polynomial.

We describe it not as CP-violation, but as CP conservation up to a background phase. So we are dealing with CP as a background symmetry.

We consider a three-Higgs-doublet model, with explicit symmetry $G_f = (\Delta(54)/Z_3) \times U(1)_{em} \times SU(2)_L$. Promoting the parameters of the fourth order potential to background fields, we have a background symmetry $G = G_b \rtimes Z_2$, with $G_b = (\Sigma(216 \times 3)/Z_3) \times U(1)_{em} \times SU(2)_L$, $G/G_f = A_4 \rtimes Z_2 \simeq S_4$ [85].

Then, we choose as source field a copy of the Higgs field with the constraint $GJ = G_b J$, thus for any J there is $h \in G_b$ such that $J = hJ_0$ with J_0 real (the CP acts as the complex conjugation here). Note that an arbitrary Higgs field does not verify $GJ = G_b J$ for this case. Such constraint may be consequence of the minimization of a particular potential as in [39] (see also Sec. 5.3) or by the field content in the Action as in [82, 83] and thus look more natural then as we are doing here, which is just choosing the source field verifying the constraint.

Following [42], we have the following doublet representations of $\Delta(54)/Z_3$ constructed from the three complex Higgs doublets ϕ_m : $(a_1, a_1^*), (a_2, a_2^*), (a_3, a_3^*)$ and (note the difference) (a_4^*, a_4) , where

$$\begin{bmatrix} a_0 \\ a_2 \\ a_2^* \end{bmatrix} = M \begin{bmatrix} \phi_1^\dagger \phi_1 \\ \phi_2^\dagger \phi_2 \\ \phi_3^\dagger \phi_3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} \phi_1^\dagger \phi_1 \\ \phi_2^\dagger \phi_2 \\ \phi_3^\dagger \phi_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_3 \\ a_4 \end{bmatrix} = M \begin{bmatrix} \phi_3^\dagger \phi_2 \\ \phi_1^\dagger \phi_3 \\ \phi_2^\dagger \phi_1 \end{bmatrix}$$

and ω is a complex number such that $\omega^3 = 1$ and $\omega + \omega^2 = -1$. Note that the only invariant tensor of the gauge group $U(1)_{em} \times SU(2)_L$ is the kronecker delta with the indices of the complex $SU(2)_L$ doublet (and algebraic combinations of the kronecker delta).

The matrix M is unitary. Also, $M^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $M^4 = 1$. So, the 9 degrees of freedom in a_n

describe fully and linearly the (at most) 9 degrees of freedom of any hermitian matrix²³, such as $\phi_j^\dagger \phi_k$.

Now we can write $\phi(\omega, a) = \phi^*(\omega^2, a^*)$ so the $\phi(\omega, a)$ is invariant under the complex conjugation (CP) $\phi \rightarrow \phi^*$ followed by the exchange of $a_n \rightarrow a_n^*$ (which comes from the exchange $\phi_2 \rightarrow \phi_3$) followed by the exchange $\omega \rightarrow \omega^2$.

Imposing that ϕ_0 is invariant under CP $\phi \rightarrow \phi^*$, then $a_3 = a_4^*$ and $a_1 = a_1^*$. The $G/G_f = S_4$ group permutes a_n (leaving a_0 invariant) and so we can have CP-violation of $h\phi_0$ due to a permutation of the a_n . Such CP-violation is however dependent on $h \rightarrow chc^{-1}$ and not on ϕ_0 which can be arbitrary (i.e. CP-invariant and non-null).

We can then use the source field $h\phi_0$ to modify the Higgs potential, adding an explicit symmetry breaking term. The modified potential constitutes an example of explicit geometrical CP-violation. Therefore, explicit geometrical CP-violation was necessarily present in the first example of spontaneous geometrical CP-violation [39]. Note that there are other examples of explicit geometrical CP-violation which do not involve the Higgs potential [82].

5.5 Generalized basis for generalized CP

Any $SU(2)_L \times U(1)_Y$ -invariant Higgs potential, either CP-violating or CP-conserving, is a strictly real potential. So, what are we really searching for when we are searching for CP-violating phases?

Consider a 3-Higgs-doublet model [64, 86]. Each Higgs doublet has 4 components. Under Z_2 given by the complex conjugation we have for the imaginary neutral component $\phi_i \rightarrow -\phi_i$ and for the imaginary charged component $\varphi_i \rightarrow -\varphi_i$. We can instead introduce a real background field ϵ with numeric value 1 or -1 , which transforms under the Z_2 as $\epsilon \rightarrow -\epsilon$. Then we can redefine the fields as $\varphi = R^\dagger \phi$ with R such that for each Higgs doublet $\phi'_i = \epsilon \phi_i$ and $\varphi'_i = \epsilon \varphi_i$. Then ϕ'_i and φ'_i is Z_2 -invariant but ϵ may appear in the parameters of the potential, since ϵ is real.

Then, a potential where ϵ is not present is CP-conserving. This may seem trivial, but the CP-transformation needs not to be the complex conjugation. It can be given by Uc where c is the CP-conjugation and U is an element of the group of background symmetries [62, 87]. The reason is the fact that group extensions are not unique [63, 88, 90].

Suppose that we start by imposing a family group H_f . Then the CP transformation Uc conserves H_f and so does $(Uc)^2$. If the CP transformation is conserved then $(Uc)^2 = UU^* \in G_f$. So, G_f contains H_f as a normal subgroup. The CP transformation Uc conserves G_f since Uc commutes with UU^* .

After we identify all the symmetries G_f of the system which commute with $U(1)_Y$, to check if CP is conserved or not we need to check if it breaks the G_f -invariants, but the CP transformation always acts on the G_f -invariants as a Z_2 transformation since $(Uc)^2 = UU^* \in$

²³Note that for the particular case of $\phi_j^\dagger \phi_k$, there are at most 6 degrees of freedom plus 2 non-negative degrees of freedom.

G_f .

So, we start with the same background field R , it is invariant under $G_g \times U(1)_{em} \times SU(2)_L$. However, under a generalized CP transformation we get the background field $R \rightarrow UR^*$. Note that $RR^\dagger = 1 = R^\dagger R$ is still left invariant by the generalized CP transformation, so we can insert $R^\dagger R$ wherever it is necessary. Also $UR^*G_f\phi = G_fUR^*\phi$ still conserves the true symmetries G_f , despite that it changes the remaining background symmetries.

Then, we change the basis of the potential using R , we call it the CP -basis. The parameters of the Higgs potential in the CP -basis (e.g. $\mathcal{Y} = R^\dagger Y R$ and $\mathcal{Z} = (R^\dagger \otimes R^\dagger)Z(R \otimes R)$) are by construction G_f -invariants. In particular, $(cU)^2 = U^*U \in G_f$ and so the CP transformation acts on the parameters of the Higgs potential as a Z_2 -transformation.

Therefore in such basis, we still need to look for terms depending on ϵ . If there are none, then CP is conserved and in that sense we would have a “real” basis. Note however that it may not be so easy to find such basis because R can be in principle any unitary matrix. But such basis always exists.

In the case of an order-4 CP transformation Uc with $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}$ [64, 86]. We have $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix}$ and $a = 1 \rightarrow 0 \rightarrow -1 \rightarrow 0$ and $b = 0 \rightarrow i \rightarrow 0 \rightarrow -i$, where the arrows indicate the action of the CP.

Then for the UU^* -invariants, we have

$$\begin{aligned} \phi_2^\dagger \phi_2 &= a^2 \varphi_2^\dagger \varphi_2 + bb^* \varphi_3^\dagger \varphi_3, \\ \phi_3^\dagger \phi_3 &= bb^* \varphi_2^\dagger \varphi_2 + a^2 \varphi_3^\dagger \varphi_3, \\ \phi_3^\dagger \phi_2 &= a^2 \varphi_3^\dagger \varphi_2 - bb^* \varphi_2^\dagger \varphi_3, \\ (\phi_1^\dagger \phi_2)^2 &= a^2 \phi_1^\dagger \varphi_2 \phi_1^\dagger \varphi_2 - bb^* \phi_1^\dagger \varphi_3 \phi_1^\dagger \varphi_3, \\ (\phi_1^\dagger \phi_3)^2 &= -bb^* \phi_1^\dagger \varphi_2 \phi_1^\dagger \varphi_2 + a^2 \phi_1^\dagger \varphi_3 \phi_1^\dagger \varphi_3, \\ (\phi_1^\dagger \phi_3)(\phi_1^\dagger \phi_2) &= a^2 \phi_1^\dagger \varphi_3 \phi_1^\dagger \varphi_2 + bb^* \phi_1^\dagger \varphi_2 \phi_1^\dagger \varphi_3, \\ (\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_1) &= a^2 \phi_1^\dagger \varphi_2 \varphi_2^\dagger \phi_1 + bb^* \phi_1^\dagger \varphi_3 \varphi_3^\dagger \phi_1, \\ (\phi_1^\dagger \phi_3)(\phi_3^\dagger \phi_1) &= bb^* \phi_1^\dagger \varphi_2 \varphi_2^\dagger \phi_1 + a^2 \phi_1^\dagger \varphi_3 \varphi_3^\dagger \phi_1, \\ (\phi_1^\dagger \phi_3)(\phi_2^\dagger \phi_1) &= a^2 \phi_1^\dagger \varphi_3 \varphi_2^\dagger \phi_1 - bb^* \phi_1^\dagger \varphi_2 \varphi_3^\dagger \phi_1 \end{aligned}$$

And also the complex conjugates. Note that $ab = 0$. We can then define $a^2 = \frac{1+\epsilon}{2}$ and $bb^* = \frac{1-\epsilon}{2}$. We can then combine the UU^* -invariants into linearly independent polynomials which are either proportional to ϵ or absent from ϵ . There is then a basis for the Higgs potential where ϵ is absent if and only if the order 4 CP transformation is conserved. Note that this does not strictly invalidate the claim that a real basis does not exist [86], because we are not using a complex notation and we are using background fields instead.

What it does show is that to deal with generalized CP it is better to treat the Higgs field as a real field and use a generalized basis involving a background field which transforms covariantly with the generalized CP transformation. The standard bases (that do not involve background fields) transform under generalized CP as if it was a standard CP transformation, thus such bases are only good to handle standard CP transformations of the Higgs fields. Therefore we believe that the model of Ref. [86] gives support to our use of background fields in the CP basis.

Note that our method requires knowledge of the G_f group of family symmetries. There are alternative methods which do not require such knowledge [89], but they are also complicated and we cannot guarantee that in a realistic situation it is not better to determine first which are the family symmetries G_f (such knowledge is required for other purposes anyway).

A generalized CP transformation is different than a Z_2 -like CP transformation only once we access the G_f -dependent degrees of freedom. Therefore, any model which displays a Z_2 -like CP transformation can have a Z_4 -like CP transformation if $G_f = Z_2$. In fact, the (background) CP symmetry of the Standard Model (without any extra degrees of freedom) may already be Z_4 -like and we have no way to know it without new experimental results [90]. This may explain why several authors believe that in the Standard Model the (background) CP symmetry is necessarily Z_2 -like [86].

6 Conclusion

Dealing with concepts which are not rigorously defined (in the mathematical sense) can have advantages with respect to an approach where every concept is rigorously defined [91]. In the context of Electroweak physics that is necessarily the case since a rigorously defined non-abelian gauge Quantum Field Theory does not exist yet. Therefore, assumptions play a key role.

But after making the usual perturbative assumptions some problems are still very complicated. That is the case of building extensions of the Standard Model²⁴, and in particular studying the Higgs potential (a symmetric polynomial of many variables [16–18]).

We should be careful: making assumptions can be used to focus on the physical questions as much as it can be used to avoid the physical questions.

To study the Higgs potential, one option is to check what are the implications of alternative assumptions. Such as non-perturbative assumptions—e.g. the ones used in lattice gauge theory or in the functional renormalization group, which can produce complementary results [27, 28]. Or working with real representations of groups—which in a real polynomial makes sense [52] and it is necessary²⁵ to deal with the approximated custodial symmetry of the Higgs potential [22, 57, 58, 73–75].

In this paper we showed that such option does lead to progress (see also the Appendix), despite that the perturbative Electroweak expansion is a good approximation to the experimental results.

One common way to define spontaneous symmetry breaking involves necessarily explicit symmetry breaking. We study Quantum Field Theories extending the Standard Model, without

²⁴Using extensions of the Standard Model is a practical way to produce predictions for experiments. But like statistical inference [92], (new) physics is not just about producing numbers. E.g. accounting all reasonable extensions, we may have one prediction for each logical possibility [93], which is a kind of look-elsewhere effect. Producing predictions where such effect is consistently accounted for is a hard problem (even if we assume spontaneous symmetry breaking of $SU(2)_L$).

²⁵also to study the physical spectrum in multi-Higgs-doublet models; to handle the pseudo-goldstone bosons in multi-Higgs models; or to do lattice simulations of the Higgs sector.

anomalies.

Defining the spontaneous breaking of a global symmetry as a particular case of explicit symmetry breaking has the advantage that the classical Action indicates which symmetries are allowed to be spontaneously broken, without making non-perturbative assumptions that such symmetries are in fact spontaneously broken, and so such Action can be used in non-perturbative studies which may give support or not to such assumptions.

We studied background fields and background symmetries: these are symmetries that despite they are already explicitly broken, can be still spontaneously broken.

We clarified the assumptions under which a recent conjecture relating spontaneous and explicit CP-violation (charge-parity) is true/false; we related explicit and spontaneous geometrical CP-violation (charge-parity).

We treated here the fields as real representations of the group of symmetries and showed that this is consistent in Quantum Field Theory. If we know the (non-CP) symmetries of the Action, then there is a basis where CP is conserved if and only if the Action is Z_2 symmetric (i.e. there are no “phases” in the Action).

While there is no reason to study the Higgs potential based on the usual perturbative assumptions only, most results on the Higgs potential obtained under the usual perturbative assumptions in the literature involve global symmetries and so are still consistent within a non-perturbative framework, since it is possible to study the Higgs potential without assuming that the local gauge $SU(2)_L$ symmetry is spontaneously broken or not, as we have shown. So our results are good both for the people who prefer a perturbative framework, as well as for the people who prefer a non-perturbative framework, who can work together or at least obtain results which are complementary, improving the understanding of extended Higgs sectors.

In conclusion, assuming gauge symmetry breaking or using only complex representations of groups is not sufficient to study the phenomenology of extended Higgs sectors.

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A No free lunch theorem

In this section we apply a well know theorem in numerical analysis to the High Energy Physics context: the “no free lunch” theorem [94–96].

In the context of search and optimization problems, the “no free lunch” theorem states that over a very general class of problems, any algorithm is on average as good as any other, including random search. There are also other classes of problems where the theorem does not hold [97], but to know if we are in such class requires anyway information about the problem we want to solve.

That means that if we do research in Quantum Mechanics with some optimization algorithm that works well for Quantum Mechanics, when we apply such algorithm to Quantum Field Theory, such algorithm can be in principle worse than random search.

If we do research in Electroweak theory with some optimization algorithm that works well before the Higgs boson was discovered, when we apply such algorithm to precision Higgs physics, such algorithm can be in principle worse than random search.

If we do research in the two-Higgs-doublet model with some optimization algorithm that works well for the two-Higgs-doublet model, when we apply such algorithm to the three-Higgs-doublet model, such algorithm can be in principle worse than random search.

And so on. The bottom line is that the algorithms we use should change accordingly to the problem. And all information about the properties of the specific problem we want to solve can make a huge difference.

This is not a question about whether we should do research in philosophy of physics, mathematical physics or phenomenology²⁶. Whether we choose to do research in philosophy of physics, mathematical physics or phenomenology, any information which is known and available about the problem we want to solve can make a huge difference. In particular, the differences between complex and real vector spaces are known and available; and spontaneous breakdown of local gauge symmetry is so far an assumption valid in perturbation theory but not in other non-perturbative methods. These informations are relevant for people doing either philosophy of physics, mathematical physics or phenomenology (as we have seen here for the phenomenology case).

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²⁶The context of the theorem is in fact numerical analysis. For instance, Feynman was not too much interested in philosophy of physics or in mathematical physics but still he argued that it was important know the properties of a problem independently of which algorithm we prefer to use (using the example of Mayan astronomers[98]). And by the way a personal note: I have always worked hard towards the goal of knowing to compute perturbative predictions from a realistic quantum model and to do realistic experimental data analyses in the context of High Energy Physics.

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